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SURFACES WHICH CAN BE GENERATED IN MORE THAN ONE WAY BY THE MOTION OF AN INVARIABLE CURVE.*

BY LUTHER PFAHLER EISENHART.

The generation of a surface by the motion of an invariable curve is an intuitive conception. When the moving curve is a straight line, we have a ruled surface. When the motion is a rotation, a curve in a plane through the axis of rotation describes a surface of revolution. To this class of surfaces belong the sphere, the anchor ring and the quadrics of revolution. Another simple type is afforded by a surface generated by the translation of a curve C; it is called a surface of translation. The various curves described by points of C are congruent; any two of them can be brought into coincidence by translation. Hence a surface of translation possesses the property that it can be generated in two ways by the motion (in this case translations) of an invariable curve. Hyperboloids of one sheet and hyperbolic paraboloids are the only doubly ruled surfaces; on any one of them there lie two one-parameter families of real straight lines such that each line of one family meets all the lines of the other family, but none of its own family. Hence they also can be generated in two ways by the motion of a straight line. In this paper we discuss other types, not so simple nor so well known, of surfaces which can be generated in more than one way by the motion of an invariable curve.

1. **Double surfaces of translation.** An important example of surfaces generated in four ways by the motion of an invariable curve is afforded by the surfaces which are surfaces of translation in two ways. The equations of such a surface S are necessarily of the forms

$$\begin{aligned} x &= X_1(x_1) + X_2(x_2) = -X_3(x_3) - X_4(x_4), \\ y &= Y_1(x_1) + Y_2(x_2) = -Y_3(x_3) - Y_4(x_4), \\ z &= Z_1(x_1) + Z_2(x_2) = -Z_3(x_3) - Z_4(x_4), \end{aligned}$$

where x_1, x_2, x_3, x_4 are parameters. Incidentally we remark that when such a surface S is known, two others, S', S'', are known also; their equations are

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* Address of the Vice-president and Chairman of Section A—Mathematics and Astronomy—American Association for the Advancement of Science, Pittsburgh, December 29, 1917.

$$x' = X_1(x_1) + X_3(x_3) = -X_2(x_2) - X_4(x_4),$$
$$y' = Y_1(x_1) + Y_3(x_3) = -Y_2(x_2) - Y_4(x_4),$$
$$z' = Z_1(x_1) + Z_3(x_3) = -Z_2(x_2) - Z_4(x_4),$$

and

$$x'' = X_1(x_1) + X_4(x_4) = -X_2(x_2) - X_3(x_3),$$
$$y'' = Y_1(x_1) + Y_4(x_4) = -Y_2(x_2) - Y_3(x_3),$$
$$z'' = Z_1(x_1) + Z_4(x_4) = -Z_2(x_2) - Z_3(x_3).$$

Lie* proposed the problem of finding all surfaces of this kind and obtained the general solution with the aid of Abel's theorem. Recently Darboux† obtained the same result in a simple and direct method without the use of Abel's theorem, which theorem follows therefore as a corollary. If $F(x, y) = 0$ defines any plane curve of the fourth order, and x_1, x_2, x_3 and x_4 are the abscissas of the points of intersection of the curve by a movable line, the functions X_i, Y_i, Z_i $(i = 1, 2, 3, 4)$ are given by

$$dX_i = \frac{dx_i}{\frac{\partial F}{\partial y_i}}, \qquad dY_i = \frac{x_i dx_i}{\frac{\partial F}{\partial y_i}}, \qquad dZ_i = \frac{y_i dx_i}{\frac{\partial F}{\partial y_i}}.$$

Hence X_i, Y_i and Z_i are the abelian integrals of the first kind relative to the four points of the curve situated on the same variable straight line.

We refer the reader to Darboux's paper for details, and content ourselves with the statement of the following additional properties. Thus Darboux has shown‡ that the tangents to the four curves C_i of coördinates (X_i, Y_i, Z_i) are parallel to the corresponding tangent planes to the surfaces S, S' and S''. He reëstablishes also the following theorem of Lie:

The lines drawn through a fixed point parallel to the tangents to any one of the curves C_i are the generators of an algebraic cone of the fourth order.

Lie§ considered the case when the quartic $F(x, y) = 0$ is decomposable into two conics. In this case one of the three surfaces, S, S' and S'', can be generated in an infinity of ways by the translation of a curve. Of this type are the minimal surface of Scherk, $e^z = \sin x \csc y$, and the minimal helicoid, $z = \tan^{-1}(y/x)$.

2. **Surfaces of Gambier.** Consider two right circular cylinders, F and R, of radii f and r respectively, placed with their axes vertical and in contact

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* Archiv for Mathematik og Naturvidenskab. Also Poincaré, Journal de Mathématiques pures et appliquées, Ser. 5, vol. 1 (1895), pp. 219–314, and Bulletin de la Société mathématique de France, vol. 29 (1901), pp. 61–86.

† Leçons des Surfaces, Second Edition, Paris, 1914, pp. 151–161.

‡ L. c.

§ Geometrie der Berührungstransformationen, Leipzig, 1896, p. 409.

along a generator of each, internally or externally. Imagine a third cylinder A of radius a attached to R so that both A and R have a common axis. As R rolls, without slipping, over the fixed cylinder F, each point of A describes an epitrochoid or a hypotrochoid, according as R is exterior or interior to F. As the argument is the same in either case, we assume that R is exterior to F. Then all the epitrochoids are congruent curves. In fact, any one of them can be brought into coincidence with any other by a translation along the axis of F and a suitable rotation about this axis. If the axis of F is taken for the z-axis of coördinates, the projection of any one of these curves on the plane $z = 0$ is defined by equations of the form

$$\begin{aligned} x &= (f + r)\cos\varphi - a\cos\left(\frac{f + r}{r}\varphi\right), \\ y &= (f + r)\sin\varphi - a\sin\left(\frac{f + r}{r}\varphi\right), \end{aligned} \tag{1}$$

where φ is the same for the curves described by points on the same generator of A. If we take any curve C on A, this curve describes a surface S upon which lies a family of congruent epitrochoids. Consequently S can be generated also by a suitable displacement of one of the latter curves, and therefore S is a surface of the type sought. These surfaces have been considered by Gambier.* We call them the *surfaces of Gambier.*

The epitrochoids considered above can be generated in a second manner, namely by taking three new cylinders F_1, R_1 and A_1, associated in a manner similar to the above, with their corresponding radii given by†

$$f_1 = \frac{af}{r}, \qquad r_1 = \frac{a}{r}(f + r), \qquad a_1 = f + r. \tag{2}$$

The one-parameter family of epitrochoids lying on a surface of Gambier generated as above are described in like manner by the points of a curve C_1 lying on the cylinder A_1 as the latter moves with R_1 when it rolls over F_1. Hence we have the following theorem of Gambier:

A surface of Gambier can be generated in three different ways by the motion of an invariable curve.

When, in particular, $r = \frac{1}{2}f$ and $a = r$, the epitrochoids are straight lines passing through the axis of F. In this case the surface is a right conoid whose particular form is determined by C. Thus any right conoid, just like any cylinder, can be generated by the motion of a curve whose points describe straight lines. We considered the converse problem in a recent paper* and established by analytical processes the theorem:

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* Comptes Rendus, vol. 158 (1914), p. 1155.

† Cf. Loria, Spezielle Algebraische und Transcendente Ebene Kurven, Leipzig (1902), p. 483.

Cylinders and right conoids are the only ruled surfaces which can be generated by the motion of an invariable curve such that points of the curve describe generators in the motion; for the conoids these curves are sections of the surface by cylinders having the axis of the conoid for a generator.

The surfaces of Gambier afford an example of surfaces generated by the motion of an invariable curve whose points describe congruent curves. So far as we know, it has never been determined whether these are the only surfaces possessing this property.

When, as in the case of the right conoids, the ratio f/r is commensurable, the motion of the cylinders is algebraic (as follows from (1)) and an infinite number of algebraic surfaces can be generated. Gambier mentions, in particular, the case when $f = 2r$, $a \neq r$, and C is an ellipse. The surface is unicursal of the sixth degree, and is defined by

$$x = r \cos \varphi + a \cos \varphi_1, \qquad y = r \sin \varphi + a \sin \varphi_1,$$

$$z = l \sin (\varphi + \varphi_1) + l' \cos (\varphi + \varphi_1),$$

where l and l' are constants. The three series of equal curves are defined by $\varphi = \text{const.}$, $\varphi_1 = \text{const.}$, $\varphi + \varphi_1 = \text{const.}$; all are ellipses.

3. **The Bennett mechanism.** In the generation of certain surfaces of the kind sought use can be made of a simple new mechanism discovered in 1903 by G. T. Bennett of Emanuel College, Cambridge.† It consists of two pairs of equal rods, of lengths a and b. They are made into a skew quadrilateral $ABCD$ with

$$AB = CD = a, \qquad BC = AD = b.$$

The rods are fastened together by hinges at the vertices, and are perpendicular to the axis of the hinge at the point of junction, the axes of the hinges not being parallel but subject to the condition that the axes at A and C made the angles α and β respectively with the axis at B. Since the angles between the rods at A and C are equal, and likewise at B and D, we put

$$\theta = DAB = BCD, \qquad \varphi = ABC = CDA.$$

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* Rendiconti del Circolo Matematico di Palermo, vol. 41 (1916), pp. 94–102.

† Engineering, vol. 76 (1903), p. 777. Darboux studied deformations of this skew quadrilateral in his memoir, Sur un problème relatif à la théorie des courbes gauches, Mémoires de l'Académie des Sciences, 1908. Borel also dealt with these quadrilaterals in his Mémoire sur les déplacements à trajectoires sphériques, Mémoires présentés par divers savants à l'Académie des Sciences, vol. 23 (1905). See also Bricard, Journal de l'École Polytechnique, Ser. 2, Cahier 11 (1906), pp. 1–93.

We remark that the axes at the extremities of any side lie in parallel planes normal to the side.

In order to find the relations between the quantities a, b, α, β, θ and φ, we express the conditions that the sum of the projections of the sides of the quadrilateral on each of three non-coplanar lines is zero. We choose as these lines the sides, AB and BC, and the axis at B, which evidently is normal to the plane ABC. In order to carry out this process more readily, we replace CD by its three components $a \cos \theta$, $a \sin \theta \cos \beta$, $a \sin \theta \sin \beta$, along BC, the perpendicular to BC in the plane ABC at C and the normal to ABC at C respectively. In like manner we replace AD by its components $b \cos \theta$, $b \sin \theta \cos \alpha$, $b \sin \theta \sin \alpha$ along AB, the perpendicular to AB in the plane ABC at A and the normal to ABC at A respectively. Then the conditions that the sum of the projections of the sides of the quadrilateral on the lines AB, BC and the axis at B are zero assume the respective forms

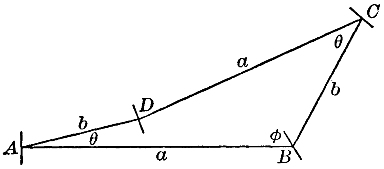

FIG. 1.

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(3)\qquad
\begin{aligned}
a - b \cos \varphi &= \cos \theta \,(b - a \cos \varphi) + a \sin \varphi \sin \theta \cos \beta,\\
b - a \cos \varphi &= \cos \theta \,(a - b \cos \varphi) + b \sin \varphi \sin \theta \cos \alpha,\\
a \sin \beta &= b \sin \alpha.
\end{aligned}
$$

The last of these equations gives the only condition upon a, b, α and β. If $\sin \theta \sin \varphi$ is eliminated from the first two of the equations (3), the result is reducible to

$$
(4)\qquad \tan \frac{\theta}{2} \tan \frac{\varphi}{2} = \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} \equiv k.
$$

Again, if a and b are eliminated from the first two of (3), the resulting equation is reducible to (4). Hence the three equations (3) are equivalent to their last and (4).

If we consider the sides BC and CD after the manner of AB and BC, we find that, in consequence of the third of (3), the axes at the extremities of CD make the angle α. In like manner the angle between the axes at the extremities of DA is β. If the angle between the planes determined by a side of the quadrilateral and the two vertices not on the side is called a *dihedral angle* of the quadrilateral, from the preceding result and the third of (3) follows the theorem:

If one dihedral angle of a skew quadrilateral with opposite sides equal remains constant in the deformation of the quadrilateral, all the dihedral angles remain constant.

These results are true for any skew quadrilateral having opposite sides equal. For the Bennett mechanism α and β are constant, and consequently k also. Since θ and φ vary in accordance with the relation (4), the ratio of the angular velocities of two opposite rods relative to either of the other two rods is found by differentiating (4) with respect to t. This gives

$$\frac{\frac{d\theta}{dt}}{\frac{d\varphi}{dt}} = -\frac{\sin\theta}{\sin\varphi} = \frac{\cos\theta+1}{\cos\varphi-1}k = \frac{\cos\theta-1}{\cos\varphi+1}\frac{1}{k}. \tag{5}$$

Bennett has shown that the mutual rotation of the rods is illustrated by the rotation of the focal radii of an ellipse of eccentricity

$$e = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}. \tag{6}$$

In fact, using the polar equation of the ellipse in the forms

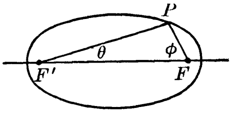

FIG. 2.

$$r_1 = \frac{l}{1-e\cos\theta}, \qquad r_2 = \frac{l}{1-e\cos\varphi},$$

and $r_1 + r_2 = 2a$, we get

$$\frac{2}{1-e^2} = \frac{1}{1-e\cos\theta} + \frac{1}{1-e\cos\varphi}.$$

From this we find

$$\tan\frac{\theta}{2}\tan\frac{\varphi}{2} = \frac{1-e}{1+e} = k,$$

which is in keeping with (4) and (6).

From (4) it is seen that when θ or φ is zero, the other is π. In these cases, as follows from (5), the ratio of the angular velocities is greatest and least, namely k and $1/k$. The ratio is unity, when $\theta = \varphi$, that is when P is at the end of the minor axis of the ellipse.

4. **Doubly-decomposable systems of Koenigs.** Let two bodies A and C form a binary system of two parameters, that is, a system in which the relative positions of the two bodies depend upon two parameters. If there are two bodies B and D, such that the systems $\boxed{AB}$ and $\boxed{BC}$ depend upon one parameter each, both parameters being independent, and likewise the systems $\boxed{AD}$ and $\boxed{DC}$ depend upon one parameter each, the system $\boxed{AC}$ is said to be doubly decomposable, according to Koenigs.* If instead of two bodies B and D possessing this property, there are n such bodies, the system $\boxed{AC}$ is n-tuply decomposable.

Consider a point M of C. As C moves relatively to B, M describes a curve $\gamma_B{}^M$, and as B moves relatively to A this curve describes a surface $\Phi_A{}^M$. In like manner, as C moves relatively to D, M describes a curve $\gamma_D{}^M$, which curve again describes $\Phi_A{}^M$, as D moves relatively to A. Hence we have the theorem of Koenigs.†

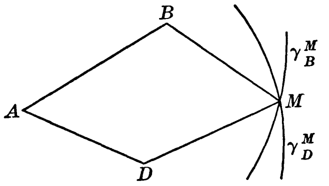

Fig. 3.

If $\boxed{AC}$ is a doubly-decomposable system, a point of C in all motions of C relative to A describes a surface which is the locus of two families of equal curves.‡

Evidently, if the system is n-tuply decomposable, the surface contains n families of equal curves.

We remark that Gambier has shown that the surfaces of Gambier can be generated in accordance with the above theorem of Koenigs. We shall not consider this case, but turn to an interesting example given by Koenigs,§ based on the Bennett mechanism.

5. **Hyperboloidal Gears.** Since the axes at the points A and B of the

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* Comptes Rendus, vol. 117 (1913), p. 989.

† L. c.

‡ The only exception arises when $\gamma_B{}^M$ and $\gamma_D{}^M$ are superposed curves without interfering with the relative motion of B and D. In this case the curves are helices, circles or straight lines.

§ L. c.

Bennett mechanism are perpendicular to AB but inclined to one another at the angle α, two bodies Σ_A and Σ_B attached to these axes can be made to revolve about these respective axes in unison by means of hyperboloidal gears.* Two such gears meet along a common generator of two hyperboloids of revolution H_1 and H_2, with the axes at A and B of the Bennett mechanism. The radii r_1 and r_2 of the circles of gorge of the hyperboloids are such that their sum or difference is a, according as the hyperboloids roll externally to one another or one is internal to the other. If θ_1 and θ_2 denote the angles made by the common generator with the axes of H_1 and H_2, then from the geometrical interrelation of H_1 and H_2 we have

$$\frac{r_1}{r_2} = \frac{\tan\theta_1}{\tan\theta_2}. \tag{7}$$

If we assume that H_1 and H_2 are exterior to one another, and project the axes on the plane normal to AB and passing through the common generator, we get the situation illustrated by Fig. 4, where C is the point of AB; L and M are projections of points on the axes of H_1 and H_2 respectively, and P is a point on the common generator. If ω_1 and ω_2 denote the angular velocities of H_1 and H_2 respectively, we have

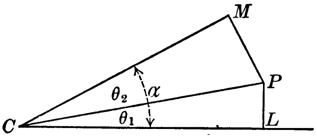

FIG. 4.

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{PL}{PM} = \frac{\omega_2}{\omega_1}. \tag{8}$$

From this relation and $\alpha = \theta_1 + \theta_2$, we derive

$$\tan\theta_1 = \frac{\sin\alpha}{\dfrac{\omega_1}{\omega_2} + \cos\alpha}, \qquad \tan\theta_2 = \frac{\sin\alpha}{\dfrac{\omega_2}{\omega_1} + \cos\alpha}.$$

Combining this result with (7) and

$$r_1 + r_2 = a, \tag{9}$$

we get

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[*] The following discussion of these gears is essentially the same as given by Reuleaux, Le Constructeur, Paris (1875), p. 463.

$$(10)\quad \frac{r_1}{\omega_2^2 + \omega_1\omega_2 \cos \alpha} = \frac{r_2}{\omega_1^2 + \omega_1\omega_2 \cos \alpha} = \frac{a}{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2 \cos \alpha}.$$

When H_1 rolls inside of H_2, in comparison with the foregoing the ratio ω_1/ω_2 is negative, and so the analogous equations are in this case

$$(11)\quad \begin{gathered} \frac{\sin \theta_1}{\sin \theta_2} = -\frac{\omega_2}{\omega_1}, \qquad \alpha = \theta_2 - \theta_1, \qquad r_2 - r_1 = a, \\ \tan \theta_1 = \frac{\sin \alpha}{\dfrac{\omega_1}{\omega_2} - \cos \alpha}, \qquad \tan \theta_2 = \frac{\sin \alpha}{-\dfrac{\omega_2}{\omega_1} + \cos \alpha}. \end{gathered}$$

6. **The Koenigs configuration.** We return to the consideration of the Bennett mechanism and place two bodies Σ_A and Σ_B on the axes at A and B, with hyperboloidal gears rolling externally to one another. We seek now to place a body Σ_C on the axis at C with a hyperboloidal gear rolling on H_2 and external to the latter. If θ_3 and r_3 are the angle and radius of the circle of gorge of the hyperboloid, H_3, of this gear, we must have

$$b = r_2 + r_3, \qquad \beta = \theta_2 + \theta_3.$$

When these values are substituted in the third of (3), the resulting equation is reducible, by means of (7) and a similar equation for r_2 and r_3, to

$$\cos \theta_1 = \cos \theta_3.$$

If $\theta_3 = \theta_1$, then $\alpha = \beta$, and consequently $a = b$. In this case we have a skew square, and by placing a gearing of angle θ_2 and radius r_2 on a body Σ_D revolving about the axis at D, we get a perfectly articulated system. The system $\Sigma_A\Sigma_C$ depends on two parameters, but the decomposition by means of Σ_B and Σ_D is similar. Hence we proceed to the other solution $\theta_3 = -\theta_1$.

This second solution leads to the case where H_3 lies within H_2. In fact, if, on the assumption $a > b$, we choose r_1 and r_2 so that

$$r_1 + r_2 = a, \qquad -r_1 + r_2 = b,$$

and take

$$\theta_1 = \frac{\alpha - \beta}{2}, \qquad \theta_2 = \frac{\alpha + \beta}{2},$$

in consequence of (3), equation (7) is satisfied. Then if we take for H_3 the hyperboloid H_1, and let it roll inside of H_2, equations analogous to (11) are satisfied. Furthermore, if we place a body Σ_D on the axis at D, and supply it with gears of the same form and size as H_2, with the understanding that the gear rolls externally to H_3 and about the gear H_1, we have a completely articulated system.

In this case the bodies $\Sigma_A \Sigma_C$ form a doubly-decomposable system, the rotation of the gears and the variation of the Bennett mechanism depending upon two independent parameters, such that Σ_B and Σ_D are the other bodies in the two decompositions. Hence a point of Σ_C, as the latter moves relative to Σ_A, generates a surface upon which there are two families of equal curves, namely skew epitrochoids.

It is readily shown that the line joining the mid-points of AC and BD is an axis of symmetry of the Bennett mechanism. It is also an axis of symmetry of the gears.

7. **Configuration of Bricard.** Another doubly-decomposable system which makes use of the Bennett mechanism has been discovered by Bricard.*

Given a triangle ABC and a point D' of its plane, and let the lengths of AD', BD' and CD' be denoted by l, m and n respectively. It is desired that a point D outside of the plane ABC be found so that in the faces ABD, ACD, BCD of the tetrahedron $ABCD$, points C', B', A' respectively can be found so that we have

$$\begin{aligned} A'D &= B'C = C'B = D'A = l, \\ A'C &= B'D = C'A = D'B = m, \qquad (12) \\ A'B &= B'A = C'D = D'C = n, \end{aligned}$$

and so that the tetrahedron $ABCD$ shall be inscribed in the tetrahedron $A'B'C'D'$, that is the point A is in the plane $B'C'D'$, B in the plane $A'C'D'$, and so on. Two tetrahedra each of which is inscribed in and circumscribed about the other were first considered by Möbius.† They are said to form a *Möbius configuration.* Bricard states, without proof, that this special Möbius configuration can be obtained as follows: Through D' draw any line L, and let α, β, γ be the points symmetric with respect to L of A, B, C respectively; then take

$$CD = A\beta = B\alpha, \qquad AD = B\gamma = C\beta, \qquad BD = A\gamma = C\alpha,$$

and construct the tetrahedron. In consequence of this choice it is possible to find points A', B', C', D' satisfying the conditions (12). The accompanying figures show how these points lie in the respective faces; their construction is unique and immediate. It remains to show that $ABCD$

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* Comptes Rendus, vol. 158 (1914), p. 110.

† Crelle, vol. 3 (1828), pp. 273–278; also Gesammelte Werke, vol. 1, pp. 405–438.

is inscribed in $A'B'C'D'$. In order to do this we make use of Möbius's general methods in the following form.

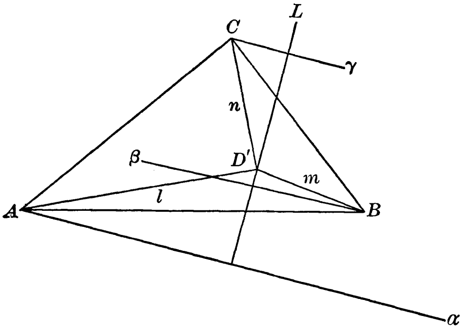

FIG. 5.

If A, B and C denote the distances of any three non-collinear points, A, B, C, from any line in their plane, and D' denotes the distance of the center of mass of three masses d_1, d_2, d_3 placed at A, B, C respectively, then

$$dD' = d_1A + d_2B + d_3C, \qquad d = d_1 + d_2 + d_3. \tag{13}$$

By taking moments about the sides of the triangle ABC, we find that D' is so placed that

$$d_1 : d_2 : d_3 = D'BC : D'CA : D'AB, \tag{14}$$

the terms on the right indicating areas of the respective triangles. It is evident then that if D' is any point in the plane of three points, A, B, C,

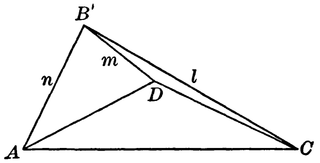

FIG. 6.

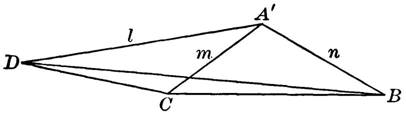

FIG. 7.

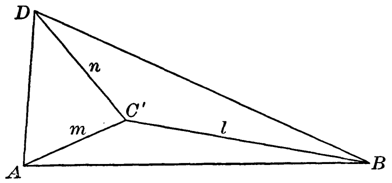

FIG. 8.

we have an equation of the form (13), it being understood that an area is positive or negative, according as the point D' lies on the same or opposite side of a side of the triangle as the interior of the latter. Hence this equation gives analytical expression to the necessary and sufficient condition that a point D' lies in the plane of A, B, C.

Applying these considerations to the tetrahedra $ABCD$ and $A'B'C'D'$, we have, in addition to (13), the following equations:

$$\begin{aligned} aA' &= a_1B + a_2C + a_3D, \\ bB' &= b_1C + b_2D + b_3A, \\ cC' &= c_1D + c_2A + c_3B. \end{aligned} \tag{15}$$

If we require that A is to lie in the plane $B'C'D'$, a similar relation must hold. We write it in the form

$$a'A = a_1'bB' + a_2'cC' + a_3'dD',$$

since we wish to avoid fractional expressions, when in the right-hand member we substitute for B', C' and D' their expressions from (13) and (15). When this substitution is made, we get

$$(a_1'b_3 + a_2'c_2 + a_3'd_1 - a')A + (a_2'c_3 + a_3'd_2)B \\ + (a_1'b_1 + a_3'd_3)C + (a_1'b_2 + a_2'c_1)D = 0.$$

Since A is not in the plane of B, C, D, we must have

$$a_2'c_3 + a_3'd_2 = 0, \qquad a_1'b_1 + a_3'd_3 = 0, \qquad a_1'b_2 + a_2'c_1 = 0. \tag{16}$$

In order that these equations be consistent, we must have

$$b_2c_3d_3 = -b_1c_1d_2. \tag{17}$$

When this is satisfied, we get the ratios $a_1' : a_2' : a_3'$ from (16).

In like manner the conditions that B, C and D lie in $A'C'D'$, $A'B'D'$ and $A'B'C'$ respectively are

$$\begin{aligned} a_2c_1d_1 &= -a_3c_2d_3, \\ a_1b_2d_1 &= -a_3b_3d_2, \\ a_1b_1c_2 &= -a_2b_3c_3. \end{aligned} \tag{18}$$

This result is due to Möbius,[*] who observed that if (17) and the second of (18) be multiplied together, we get

$$a_1b_1c_1d_1 = a_3b_3c_3d_3,$$

which follows also from the first and third of (18). Hence if three of the vertices of $ABCD$ are inscribed in $A'B'C'D'$, so also is the fourth. For a general method of finding $A'B'C'D'$ for a given $ABCD$ we refer the reader to Möbius's article, and turn now to the proof that the tetrahedra of Bricard satisfy these conditions.

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[*] L. c.

From Fig. 5 we find

$$a_1 : a_2 : a_3 = AD'\beta : -AD'\gamma : d_1,$$

$$b_1 : b_2 : b_3 = -BD'\gamma : d_2 : -AD'\beta,$$

$$c_1 : c_2 : c_3 = d_3 : AD'\gamma : BD'\gamma.$$

As these values satisfy (17) and (18), we have a configuration of Möbius.

In consequence of the relations (12), the six quadrilaterals $A'BC'D$, $B'CD'A$, $AC'BD'$, $A'CD'B$, $A'DB'C$, $AB'DC'$ are skew parallelograms. Together they form a kind of skew parallelepiped such that the three edges meeting in a vertex are coplanar. Hence Bricard calls it a *skew parallelepiped with flat trihedral angles*. If we imagine D' fixed, the possible deformations of the figure are accomplished by allowing A, B and C to move in circles of respective radii l, m and n about D' in their plane. Hence the deformation involves three parameters, when l, m and n are fixed.

If we require that $A'BC'D$ be deformed so that one of its dihedrals is constant (and consequently all are constant by the theorem of § 3), the deformations involve only two independent parameters. Consider the dihedral with edge BC'. Since C' is in the face ABD, and B is in the face $A'C'D'$, the angle of the planes ABD and $A'C'D'$ is, by hypothesis, constant in the deformation. Proceeding in like manner with the other three dihedrals of the skew parallelogram $A'BC'D$, we find that each of the following pairs of planes meet under angles constant in the deformation:

$$A'C'D',\ BCD; \qquad A'B'C',\ BCD; \qquad A'B'C',\ ABD.$$

The dihedral of the skew parallelogram $A'DB'C$ with the edge $A'D$ is the angle between $A'B'C'$ and BCD, which as we have just shown is constant. Hence all the dihedrals of $A'DB'C$ are constant. Proceeding in this way we show that *all six parallelograms of a skew parallelepiped with flat vertices have constant dihedrals, when a dihedral of any one parallelogram is constant.*

The foregoing result shows that there are deformable skew parallelepipeds whose six skew parallelograms form Bennett mechanisms. The axes of the mechanism $A'BC'D$ are the normals a', b, c', d to the faces

BCD, $A'C'D'$, ABD, $A'B'C'$ of the tetrahedra, at A', B, C' and D. In similar manner we get the axes of the other mechanisms.

An edge of a mechanism and its adjoining axes form an invariable system, and we can attach to it a body. Thus we attach to $A'D$ and its axes, a', d, a body $\Sigma_{a'd}$ and to BC' and its axes, b, c', a body $\Sigma_{bc'}$. These are articulated by means of the mechanism $A'BC'D$. By means of $D'BC'A$ we articulate $\Sigma_{ad'}$ with $\Sigma_{bc'}$. In like manner $\Sigma_{a'd}$ and $\Sigma_{ad'}$ are articulated with $\Sigma_{b'c}$ by means of $A'CB'D$ and $AB'CD'$. Hence the set $\Sigma_{a'd}$, $\Sigma_{ad'}$ forms a doubly-decomposable system. In like manner the skew parallelepiped defines five other doubly-decomposable systems, so that by this means we are able to generate surfaces of the kind sought, as described in § 4.

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